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MATHEMATICS

On the Spectrum of Curvature Operators of Conformally Flat Lie Groups with a Left-Invariant Riemannian Metric

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The spectrum of differential operators on Riemannian manifolds were studied by many authors (see survey [1] and paper [2], in which conformally flat homogeneous Riemannian spaces were studied). This paper continues [3], where the spectra of curvature operators on three-dimensional Lie groups with a left-invariant Riemannian metric were considered.

Let (M, g) be a Riemannian n-manifold with the Levi-Chivita connection ∇; given smooth vector fields X, Y, Z, and V on M, $R(X, Y)Z = [\nabla_Y, \nabla_X]Z + \nabla_{[X, Y]}Z$ is the Riemannian curvature tensor and $r(X, Y) = \mathrm{tr}(V \to R(X, V)Y)$ is the Ricci tensor; by $s = \mathrm{tr}(r)$ we denote the scalar curvature and by $A = \frac{1}{n-2}\left(r - \frac{sg}{2(n-1)}\right)$, the one-dimensional curvature tensor. The Christoffel symbols of the metric are denoted by Γ^k_{ij}.

It is known (see [4]) that

$$R = W + A \oslash g,$$

where W is the Weyl tensor and $(A \oslash g)(X, Y, Z, V) = A(X, Z)g(Y, V) + A(Y, V)g(X, Z) - A(X, V)g(Y, Z) - A(Y, Z)g(X, V)$ is the Kulkarni–Nomizu product.

The Riemannian metric g generates the inner product

$$(X_1 \wedge Y_1, X_2 \wedge Y_2) = \det(g_x(X_i, Y_j)).$$

in the fibers $\Lambda^2 T_xM$, $x \in M$.

The Riemann curvature tensor generates the curvature operator $\mathcal{R}\colon \Lambda^2 T_xM \to \Lambda^2 T_xM$ defined by $(X \wedge Y, \mathcal{R}(U \wedge V)) = R_x(X, Y, U, V)$, where $R_x(X, Y, U, V) = g_x(R(X, Y)U, V)$. We also define the Ricci operator ρ by $g(\rho(X), Y) = r(X, Y)$ and the one-dimensional curvature operator $\mathcal{A}$ by $g(\mathcal{A}(X), Y) = A(X, Y)$.

We denote the principal values of the Ricci operator by $\rho_1, \rho_2, \ldots, \rho_n$ and those of the one-dimensional curvature operator by $a_1, a_2, \ldots, a_n$; the sectional curvature in a two-dimensional direction $e_i \wedge e_j$ is denoted by $K_{ij} = K_\sigma(e_i \wedge e_j)$.

A Riemannian manifold (M, g) of dimension $n \geq 4$ is said to be conformally flat if the Weyl tensor W is trivial: $W = 0$. In what follows, we assume M to be conformally flat.

Let $x \in M$, and let T_xM be the tangent space at x. In T_xM, there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ in which the Ricci and one-dimensional curvature operators are diagonalizable (see[5]). In this basis,

$$K_{ij} = R_{ijji} = \frac{1}{n-2}\left(\rho_i + \rho_j - \frac{s}{n-1}\right);$$
$$a_i = \frac{1}{n-2}\left(\rho_i - \frac{s}{2(n-1)}\right); \quad (1)$$
$$K_{ij} = a_i + a_j;$$
$$\rho_i - \rho_j = (n-2)(a_i - a_j).$$

It is easy to see that, in the basis $\{e_i \wedge e_j\}_{i<j}$, the matrix of the curvature operator $\mathcal{R}\colon \Lambda^2 M \to \Lambda^2 M$ is diagonalizable as well, and the spectrum of $\mathcal{R}$ is $\{K_{ij}\}_{i<j}$, where the $K_{ij} = K_\sigma(e_i \wedge e_j)$ are sectional curvatures (see [6] for more details).

In what follows, we assume that (M, g) is an n-dimensional $(n \geq 4)$ Lie group with a left-invariant Riemannian metric or a metric Lie group.

Theorem 1. *The principal values of the Ricci and one-dimensional curvature operators on an n-dimensional conformally flat metric Lie group can take at most two different values. Moreover, if* $\rho_i \neq \rho_j$ *(respectively,* $a_i \neq a_j$*), then* $K_{ij} = 0$.

The proof of this theorem is based on the following lemmas.

Lemma 1. *For* $1 \leq i, j, k \leq n$ *(provided that* $\rho_i \neq \rho_j$*), the following relations hold:*

$$\Gamma^k_{ij} = -\Gamma^j_{ik},$$
$$\Gamma^j_{ij} = 0,$$

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Four-dimensional Lie algebras of conformally flat metric Lie groups: the spectrum of the curvature operator

No.	Lie algebra	spec ($\mathcal{R}$) = $\{K_{12}, K_{13}, K_{14}, K_{23}, K_{24}, K_{34}\}$
1	$4\mathbb{A}_1$	$\{0, 0, 0, 0, 0, 0\}$
2	$\mathbb{A}_{3,6} \oplus \mathbb{A}_1$	$\{0, 0, 0, 0, 0, 0\}$
3	$\mathbb{A}_{3,9} \oplus \mathbb{A}_1$	$\left\{\frac{A^2(1+M^2)}{4}, \frac{A^2(1+M^2)}{4}, 0, \frac{A^2(1+M^2)}{4}, 0, 0\right\}$
4	$\mathbb{A}_{3,3} \oplus \mathbb{A}_1$	$\{-A^2, -A^2, 0, -A^2, 0, 0\}$
5	$\mathbb{A}_{3,7}^{\alpha} \oplus \mathbb{A}_1$	$\{-\alpha^2A^2, -\alpha^2A^2, 0, -\alpha^2A^2, 0, 0\}, \alpha > 0$
6	$\mathbb{A}_{4,5}^{\alpha,\beta}, \alpha = \beta = 1$	$\{-L^2, -L^2, -L^2, -L^2, -L^2, -L^2\}$
7	$\mathbb{A}_{4,6}^{\alpha,\beta}, \alpha = \beta > 0$	$\{-\alpha^2L^2, -\alpha^2L^2, -\alpha^2L^2, -\alpha^2L^2, -\alpha^2L^2, -\alpha^2L^2\}$
8	$\mathbb{A}_{4,12}$	$\{-(A^2+B^2), -(A^2+B^2), 0, -(A^2+B^2), 0, 0\}$

$$(\rho_i - \rho_j)\Gamma^j_{ki} = (\rho_k - \rho_i)\Gamma^i_{jk},$$
$$(a_i - a_j)\Gamma^j_{ki} = (a_k - a_i)\Gamma^i_{jk}.$$

Indeed, note that the one-dimensional curvature tensor satisfies the condition $A_{ik,j} - A_{ij,k} = 0$, because the group if conformally flat. Therefore,

$$(\nabla_X\rho)(Y, Z) - \frac{1}{2(n-1)}g((\nabla_X s)Y, Z)$$
$$= (\nabla_Y\rho)(X, Z) - \frac{1}{2(n-1)}g((\nabla_Y s)X, Z)$$

and

$$(\nabla_X\rho)(Y, Z) = (\nabla_Y\rho)(X, Z).$$

Setting $X = e_i$, $Y = e_j$, and $Z = e_k$, we obtain

$$(\nabla_i\rho)_{jk} = \Gamma^k_{ij}\rho_k + \Gamma^j_{ik}\rho_j = \Gamma^k_{ij}(\rho_k - \rho_j);$$
$$(\nabla_j\rho)_{ik} = \Gamma^k_{ji}(\rho_k - \rho_i);$$
$$(\rho_k - \rho_j)\Gamma^k_{ij} = (\rho_k - \rho_i)\Gamma^k_{ji}.$$

Lemma 2. *Let (M, g) be an n-dimensional conformally flat metric Lie group, and let $a_1, a_2, \ldots a_k$ be the principal values of the one-dimensional curvature operator of multiplicities $m_1, m_2, \ldots, m_k$, respectively. Then*

$$m_2\frac{a_1 + a_2}{a_1 - a_2} + m_3\frac{a_1 + a_3}{a_1 - a_3} + \ldots + m_k\frac{a_1 + a_k}{a_1 - a_k} = 0. \qquad (2)$$

This lemma is proved by straightforward calculations using Lemma 1, the relation $K_{ij} = a_i + a_j$, and Milnor's formula

$$K_{ij} = -\frac{3}{4}\sum_{k=1}^{n}(c^k_{ij})^2 + \frac{1}{4}\sum_{k=1}^{n}(c^j_{ik})^2 + \frac{1}{4}\sum_{k=1}^{n}(c^i_{jk})^2$$
$$-\sum_{k=1}^{n}c^i_{ik}c^j_{jk} + \frac{1}{2}\sum_{k=1}^{n}c^k_{ij}(c^j_{ki} - c^i_{kj}) + \frac{1}{2}\sum_{k=1}^{n}c^i_{jk}c^j_{ik},$$
$$c^k_{ij} = g([e_i, e_j], e_k) = \Gamma^k_{ij} - \Gamma^k_{ji}$$

for sectional curvatures in an orthonormal basis [7].

Let us complete the proof of the theorem. Without loss of generality, we can assume that the a_1 is the maximum (in absolute value) principal value of the one-dimensional curvature operator. Then the assumption that there are at most two different principal values leads to a contradiction, because the terms on the left-hand side of (2) are nonnegative, and at least one of them is positive. Thus, in the case of precisely two principal values a_1 and a_2, we have $a_1 = -a_2$. This proves the theorem.

Remark 1. The converse of the theorem proved above is false. Examples are Lie groups with a left-invariant Riemannian metric and harmonic Weyl tensor which are not conformally flat, because the one-dimensional curvature tensor in this case is also the Codazzi tensor (see [4] for more details).

Corollary 1. *One of the following two cases occurs.*

Case 1. *The spectrum of the one-dimensional curvature operator consists of one eigenvalue of multiplicity n: $a_1 = \ldots = a_n = a$. In this case, the spectrum of the curvature operator consists of one eigenvalue, too. The Lie group in this case is an Einstein manifold. An example of such a group is a Lie group whose Lie algebra is the semidirect sum $R^{n-1} \dot{\oplus}_\phi R$ with respect to some homomorphism $\phi: R \to \mathrm{Der}(R^{n-1})$.*

Case 2. *The principal one-dimensional curvatures have two eigenvalues of multiplicities k and $n - k$: $a_1 = a_2 = \ldots = a_k = -a_{k+1} = \ldots = -a_n = -a$. In this case, the spectrum of the Ricci operator consists of two eigenvalues of multiplicities k and $n - k$ as well, and the spectrum of the operator curvature has the form $\{2a, \ldots, 2a, 0, \ldots, -2a, \ldots, -2a\}$. An example of such a group is a Lie group whose Lie algebra has the form*

$$su(2) \oplus (R^{n-4} \dot{\oplus}_\phi R),$$

where $\dot{\oplus}_\phi$ *denotes the semidirect sum of Lie algebras with respect to some homomorphism*

$$\phi: R \to \mathrm{Der}(R^{n-4}).$$

Let (M, g) be an oriented Riemannian 4-manifold. We define the Hodge operator as the unique isomorphism of vector spaces $*: \Lambda_x^2 M \to \Lambda_x^2 M$ such that $\alpha \wedge (*\beta) = \langle\alpha, \beta\rangle \mathrm{vol}$ for any $\alpha, \beta \in \Lambda_x^2 M$, $x \in M$, where vol denotes the volume form on M.

If the Hodge and curvature operators commute, i.e., $\mathcal{R}* = *\mathcal{R}$, then the Riemannian metric is called a Thorpe metric, and the manifold itself is called a Thorpe manifold. Such manifolds were studied in [4]; in [8], it was shown that all four-dimensional Thorpe manifolds are Einstein.

If the Hodge and curvature operators anticommute, i.e., $\mathcal{R}* = -*\mathcal{R}$, then, following [8], we refer to the Riemannian metric as an anti-Thorpe metric and to the manifold as an anti-Thorpe manifold.

Theorem 2. *Let* $\mathfrak{g}$ *be the real four-dimensional Lie algebra of a conformally flat Lie group G with a left-invariant Riemannian metric. Then, in the basis* $\{e_i \wedge e_j\}_{i<j}$, *the spectrum* $\mathrm{spec}(\mathcal{R})$ *of the curvature operator* $\mathcal{R}$ *is as presented in the table.*

Remark 2. A metric g on an oriented Riemannian 4-manifold is anti-Thorpe if and only if g is conformally flat and has scalar curvature zero (see [8]). The four-dimensional conformally flat metric Lie groups corresponding to cases 1, 2, 6, and 7 in the table are Thorpe manifolds, and the groups corresponding to cases 1 and 2 are anti-Thorpe manifolds.

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REFERENCES

1. M. A. Berger, *Panoramic View of Riemannian Geometry* (Springer-Verlag, Berlin, 2003).
2. D. V. Alekseevskii and B. N. Kimel'fel'd, Math. Notes **24** (1), 559–562 (1978).
3. D. N. Oskorbin and E. D. Rodionov, Dokl. Math. **87**, 307–309 (2013).
4. A. L. Besse, *Einstein Manifolds* (Springer-Verlag, Berlin, 1987; Mir, Moscow, 1990).
5. E. D. Rodionov and V. V. Slavskii, *Proceedings of International Congress of Mathematicians "Differential Geometry and Applications," Brno, Czech Republic, 1998* (Berlin, 1998), pp. 111–126.
6. O. P. Gladunova, E. D. Rodionov, and V. V. Slavskii, Vladikavkaz. Mat. Zh., No. 3, 3–16 (2011).
7. J. Milnor, Adv. Math. **21**, 293–329 (1976).
8. D. Kim and Kh. Kim, Siberian Math. J. **44** (5), 817–820 (2003).

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